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LETTER TO THE EDITOR

Supersymmetric quantum mechanics, anomalies and factorization

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Abstract. The explicit symmetry breaking phenomenon in supersymmetric quantum mechanics (SUSY QM) is analysed in terms of anomalies. The anomalous behaviour can be assigned to the supersymmetric charges which define the two Hamiltonians of the model. The relation between the presence of anomalies and the factorization of the Schrödinger equations is also discussed.

It is not necessary to remark on the important role that supersymmetry, a special kind of symmetry which connects both bosonic and fermionic degrees of freedom, plays in most branches of theoretical physics. In particular, supersymmetric quantum mechanics (SUSY QM), the simplest framework that exhibits the aforementioned Bose–Fermi symmetry, constitutes in its own right the ideal scenario in order to analyse the supersymmetry breaking phenomenon. Unless otherwise noted we restrict ourselves in the following to one-dimensional problems. Whenever the superpotential function has no singularities in the whole domain $-\infty < x < \infty$, it happens that SUSY QM models can be classified into the cases where supersymmetry appears either unbroken or spontaneously broken. To be more specific, if SUSY remains unbroken the energy of the non-degenerate ground state is zero while the spontaneous breaking translates into a degenerate ground state with positive energy (see below) [1]. In any case, two other realizations of supersymmetry have been considered in the literature, namely dynamical or explicit breakings. With regard to the first case we simply point out that the most significant results appear to be associated with the semiclassical instanton method [2]. Going to the second case, Jevicki and Rodrigues were the first to realize that singular superpotentials yield models which exhibit unexpected features such as the emergence of eigenstates with negative energy and the unpairing of bosonic and fermionic states with positive energy [3]. The purpose of this letter is twofold:

(i) To describe the explicit breaking of supersymmetry in terms of the anomalous behaviour of the charges which define the SUSY partner Hamiltonians.

(ii) The presence of anomalies enables us to consider a new form of factorizing Schrödinger equations. Qualitatively speaking, the factorization procedure introduces a reference level over the ground state. In such a case, it is the anomaly that allows us to recover the whole spectrum.

To begin with let us repeat, mainly to set the notation, some important features about one-dimensional SUSY QM [4]. In its simplest formulation the model is specified

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by a pair of Hamiltonians

$$H_s = \begin{pmatrix} H & 0 \\ 0^+ & H_- \end{pmatrix}. \tag{1}$$

If we introduce the linear differential operators Q and Q_+ in terms of the superpotential function $W(x)$ as follows

$$Q = \frac{\mathrm{d}}{\mathrm{d}x} + W(x) \qquad q_+ = -\frac{\mathrm{d}}{\mathrm{d}x} + W(x) \tag{2}$$

then H_- and H_+ are given by [4]

$$H_- = Q_+ Q \qquad H_+ = QQ_+. \tag{3}$$

In other words

$$H_\mp = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_\pm(x) \tag{4}$$

where

$$V_\pm(x) = W^2(x) \pm W'(x) \tag{5}$$

the prime being a derivative with respect to x. The eigenfunctions of H_s are written in its two-component form, namely

$$|\Psi(x)\rangle = \begin{pmatrix} \Psi_+(x) \\ \Psi_-(x) \end{pmatrix}. \tag{6}$$

With regard to the hypothetical zero-energy eigenstates $|\Psi_0(x)\rangle$, it occurs that the supercharges annihilate them, that is

$$Q\Psi_{0-}(x) = Q_+\Psi_{0+}(x) = 0 \tag{7}$$

so that they can be obtained by solving first-order differential equations. Simply put

$$\Psi_{0\pm}(x) \sim \exp\left(\pm\int^x W(y)\,\mathrm{d}y\right). \tag{8}$$

As square integrable zero-energy modes demand boundary conditions like

$$\Psi_\pm(x) \to 0 \qquad \text{as } x \to \pm\infty \tag{9}$$

(8) says that the zero-mode, if it exists, must be non-degenerate. In such a case, its logarithmic derivative provides us with the superpotential $W(x)$. It may be interesting at this point to restate the two main properties of SUSY QM which are taken for granted.

(i) The spectrum of H_s is bounded from below by zero. (For the sake of simplicity we restrict ourselves in the following to models exhibiting only discrete spectra. In any case the extension of our arguments to cases including both discrete and continuous spectra can be performed without special difficulty). Let us consider Ψ_i $(i = -, +)$ eigenstates of H_i with eigenvalues E_i. As the Hamiltonians factorize according to the form exposed in (3), we have

$$\langle\Psi_i|H_i|\Psi_i\rangle = \|A_i|\Psi_i\rangle\|^2 \qquad A_- = Q \qquad A_+ = Q_+. \tag{10}$$

Whenever the conditions $Q\dagger = Q_+$ and $(Q_+)\dagger = Q$ are fulfilled. In such a case the eigenstates E_i can either be positive or vanish (if the energy is zero we recover (7)).

(ii) The strictly positive eigenvalues E_i appear in pairs. Precisely, by considering again Ψ_i it happens that $A_i\Psi_i$ ($i = -$ or $+$) is an eigenstate of H_i ($i = +$ or $-$) with the same eigenvalue.

However the preceding picture changes dramatically when considering singular superpotentials. Such SUSY QM models exhibit explicit breaking so that unexpected features like negative energy eigenstates or unpairing of states with non-zero energy are feasible. Two different interpretations of the phenomenon have been considered in the literature:

(i) The SUSY algebra holds as an algebra of formal differential operators but not as an operator but not as an operator algebra in Hilbert space [5].

(ii) The presence of 'pathological' states without superpartner means no violation of SUSY but supersymmetry itself is realized on a narrowed class of states [6].

Next we establish the connection between SUSY QM explicitly broken and anomalies. Remembering that anomalies appear whenever a symmetry at classical level is not preserved when going to the quantum realm, the point is that an operator becomes anomalous if it does not keep invariant the domain of definition of the Hamiltonian. In such a case, the Ehrenfest theorem is broken and the classical equation of motion loses its validity if we go to the quantum level and take expectation values [7]. Returning to SUSY QM, the description of the explicit breaking in terms of anomalies is transparent: it is the supercharges Q and Q_+ themselves that exhibit the anomalous behavour. D_{H-} and D_{H+} being the domains of definition of the self-adjoint operators H_- and H_+ respectively, the action of supersymmetry on $|\Psi\rangle$ translates into

$$\begin{pmatrix} 0 & Q \\ Q_+ & 0 \end{pmatrix}\begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix} = \begin{pmatrix} Q\Psi_- \\ Q_+\Psi_+ \end{pmatrix} \tag{11}$$

and the anomaly appears if $QD_{H-} \not\subset D_{H+}$ or $Q_+D_{H+} \not\subset D_{H-}$.

Next let us consider a superpotential $W(x)$ of the type

$$W(x) = x - \frac{1}{x} \tag{12}$$

which yields Hamiltonians like

$$H_- = -\frac{d^2}{dx^2} + x^2 - 3 \tag{13a}$$

$$H_+ = -\frac{d^2}{dx^2} + x^2 \frac{2}{x^2} - 1. \tag{13b}$$

As (13a) is the equation of the harmonic oscillator, it happens that H_- becomes essentially self-adjoint in $D_{H-} = C_0^\infty(R, dx)$. In addition the eigenfunctions and eigenvalues will read

$$\Psi_{n-} \sim \exp(-x^2/2) H_n(x) \tag{14a}$$

$$E_{n-} = 2(n-1) \qquad n = 0, 1, \ldots \tag{14b}$$

where $H_n(x)$ represent, as usual, Hermite polynomials. In such a case, we can split the domain of H_- into its even and odd parts

$$D_{H-\text{even}} = \{\Psi \varepsilon D_{H-} / \Psi(x) = \Psi(-x)\} \tag{15a}$$

$$D_{H-\text{odd}} = \{\Psi \varepsilon D_{H-} / \Psi(x) = -\Psi(-x)\}. \tag{15b}$$

Going to (13*b*) we need a more detailed analysis. H_+ represents a particular case of the Hamiltonian

$$h = -\frac{d^2}{dx^2} + x^2 + \frac{\lambda}{x^2} \tag{16}$$

which has been carefully considered in the literature. It happens that $\lambda_c = 3/8$ points out the dividing line between the cases where general self-adjoint extensions are necessary and those in which the extension is unique [8]. Above the critical point λ_c, at $\lambda = 2$, we can start from $D_h = C_0^\infty(R - \{0\}, dx)$. In these conditions it is not difficult to verify that $d_- = d_+ = 0$ (being d_i, $i = -, +$, the deficiency indices associated with (16)). From a mathematical point of view as we move on the limit point the wavefunctions must vanish at $x = 0$ and h is essentially self-adjoint on D_h. Returning to (13*b*), we must consider the so-called Friedrichs extension so that H_+ becomes self-adjoint with domain

$$D_{H+}\left\{\Psi \varepsilon L^2(R, dx) \text{ with } \left[-\Psi'' + \left(x^2 + \frac{2}{x^2} - 1\right)\Psi\right] \varepsilon L^2(R, dx) \text{ and } \Psi(0) = 0\right\} \tag{17}$$

Expressed in more physical terms, the singularity acts as an impenetrable barrier. This effect is tantamount to the compactification of the space so that the system lives only on the half-line. However, the one-dimensional non-degeneracy theorem is not necessarily valid for a potential with singular points [9] and we can construct out wavefunctions which are complete on the whole line. The price we pay is the double degeneracy (even the ground state) of the spectrum. Following the conventional procedure [10, 11] we obtain

$$\Psi_{n+\text{even}} \sim \{x^2 \exp(-x^2/2) M_n(-n, 5/2, x^2) \qquad -\infty < x < \infty\} \tag{18a}$$

$$\Psi_{n+\text{odd}} \sim \begin{cases} \{x^2 \exp(-x^2/2) M_n(-n, 5/2, x^2) & x \geqslant 0\} \\ \{-x^2 \exp(-x^2/2) M_n, 5/2, x^2) & x < 0\} \end{cases} \tag{18b}$$

where M_n are Kummer's confluent hypergeometric functions. Both eigenstates (18*a*) and (18*b*) have the same energy:

$$E_{n+} = 4(n+1) \qquad n = 0, 1, \ldots \tag{19}$$

so that the domain of H_+ also splits into even and odd parts

$$D_{H+\text{even}} = \{\Psi \varepsilon D_{H+} / \Psi(x) = \Psi(-x)\} \tag{20a}$$

$$D_{H+\text{odd}} = \{\Psi \varepsilon D_{H+} / \Psi(x) = -\Psi(-x)\}. \tag{20b}$$

In light of equations (15) and (20), the scheme in which the anomaly appears would be

$$QD_{H-\text{odd}} \subset D_{H+\text{even}} \qquad \text{and} \qquad Q_+ D_{H+\text{even}} \subset D_{H-\text{odd}} \tag{21a}$$

$$QD_{H-\text{even}} \not\subset D_{H+\text{odd}} \qquad \text{and} \qquad Q_+ D_{H+\text{odd}} \not\subset D_{H-\text{even}} \tag{21b}$$

so that the realization of SUSY is tantamount to a double explicit breaking. In effect, starting from $D_{H-\text{even}}$ we get non-square integrable wavefunctions. On the other hand, once the odd extension $D_{H+\text{odd}}$ is performed we cannot come back to D_{H-} by acting with Q_+.

In addition, the presence of anomalies has important consequences when considering the so-called method of factorization. It has been known for a long time that Schrödinger-like equations can be written as products of a pair of linear differential operators [12]. As a matter of fact it is supersymmetry that enables us to factorize the Hamiltonian [13]. If we have

$$H_1 = -\frac{d^2}{dx^2} + V(x) \tag{22}$$

it happens that H_1 can be always be factorized in the form

$$H_1 = A_{1-}A_{1+} + \varepsilon_1 \tag{23a}$$

$$A_{1\pm} = \pm\frac{d}{dx} + U_1(x) \tag{23b}$$

whenever the function $U_1(x)$ satisfies

$$U_1^2 - U_1' = V - \varepsilon_1. \tag{24}$$

If A_{1-} is the adjoint of A_{1+}, $A_{1-}A_{1+}$ represents a positive semi-definite operator so that the factorization is meaningful only if ε_1 fulfils $\varepsilon_1 \leqslant E_0$, E_0 being the energy of the ground state $\Psi_0(x)$. To be precise, if $\varepsilon_1 = E_0$, the condition (24) is clearly satisfied if

$$U_1(x) = -\frac{\Psi_0'}{\Psi_0}. \tag{25}$$

Returning to the conventional notation, $A_{1+}(A_{1-})$ are the supercharges $Q(Q_+)$, $U_1(x)$ represents the superpotential $W(x)$ while the ground state Ψ_0 of H_1 corresponds to the zero-mode of Q_+Q. Next we can generate the partner Hamiltonian

$$H_2 = A_{1+}A_{1-} + \varepsilon_1 \tag{26}$$

so that H_2 has the same spectrum of eigenvalues as H_1 except for missing the ground state. One then keeps iterating this method, thus obtaining a hierarchy of Hamiltonians. In order not to clutter the article we recommend the readers to go through the work of Sukumar, where the possibility of factorization with energy ε_1 below the ground state has been analysed in detail [14].

Now we deal briefly with the relationship between explicit breaking of SUSY, anomalies and factorization. If $A_{1-}A_{1+}$ is no longer a positive semi-definite operator, i.e. A_{1-} does not represent the adjoint of A_{1+}, the factorization of H_1 is feasible with ε_1 above the ground state energy E_0. In such a case the negative eigenvalues of the operator $A_{1-}A_{1+}$ enable us to recover the whole spectrum of H_1. On the other hand, the Hamiltonians we obtain via SUSY lose their isospectral character with respect to their partners so that the concept of hierarchy acquires a different meaning.

To finish we would like to stress the following points:

(i) The potential of equation (12) represents a particular case of

$$W(x) = x - \frac{g}{x} \tag{27}$$

a model where the different realizations of SUSY (unbroken, spontaneously broken or explicitly broken) can be analysed in terms of the coupling constant g.

(ii) In a recent paper [15] we introduced a family of one-dimensional SUSY QM models which exhibit explicit breaking. As they include both discrete and continuous spectra it would be interesting to study the realization of the Levinson theorem in its SUSY version.

(iii) With regard to factorization, the presence of anomalies breaks the isospectral property of the hierarchy of hamiltonians. As the relation between SUSY, the 'shape invariance' of the hierarchy of Hamiltonians and the solvability of the first member of the series is a well known fact [16], it would be worthwhile to discuss the effect of the SUSY explicit breaking. We shall report on these subjects elsewhere.

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